

BOUNDARIES OF ATTRACTORS AS OMEGA LIMIT SETS

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When a system of ordinary differential equations on \mathbb{R}^d have an attractor \mathcal{A} given as the omega-limit set of a compact set D [$\mathcal{A} = \Omega_D$], then provided that D contains \mathcal{A} necessarily $\partial\mathcal{A} = \Omega_{\partial D}$, i.e. the boundary of the attractor is the omega-limit set of the boundary of D . In general only a weaker result, $\partial\mathcal{A} \subseteq \Omega_{\partial D}$, can be shown to hold for random attractors arising from systems of stochastic ordinary differential equations, where now D is any compact set such that \mathcal{A} is contained in D with a positive probability. However, if in addition $\mathcal{A} \cap D$ is empty with positive probability, then in fact $\mathcal{A} = \Omega_{\partial D}$.

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Introduction

The globally attracting sets that play an important rôle in the study of the asymptotic behaviour of both deterministic and stochastic dynamical systems are usually constructed, at least theoretically, as the omega limit sets of a suitable set (or collection of sets). This omega-limit set construction roughly corresponds to the intuitive picture of a large ball of initial conditions (D) evolving under the flow and “contracting down” onto the attractor. Such an intuition suggests that the boundary of the attractor will consist precisely of the omega limit set of the boundary of the original set (∂D).

In this paper we give a simple proof of this for the deterministic case, and discuss related results for random attractors. While we believe that this result is of independent theoretical interest, our main motivation was the design of an efficient numerical method to compute random attractors in planar stochastic ODEs. This application is presented by Tearne in [15].

1. Preliminaries

The results in this paper concern the global attractors of deterministic and random dynamical systems on \mathfrak{R}^d . Although we always have in mind the application of these results to systems of (possibly stochastic) ordinary differential equations, the assumptions are naturally expressed in the framework of abstract semiflows (or cocycles), and hence the results are applicable to a wider class of examples.

In [15] similar results are obtained under the assumption that the dynamical systems considered are defined for all values of t , i.e. that solutions can be guaranteed to exist for all negative, as well as all positive times. In this paper we only assume that solutions exist for some (possibly short) negative time interval. The key to this generalisation is the following lemma, based on Brouwer's theorem on the Invariance of Domain^a: essentially it guarantees that continuous injective maps on \mathfrak{R}^d behave (for our purposes) like continuous invertible maps.

Lemma 1.1. *Let $f: \mathfrak{R}^d \rightarrow \mathfrak{R}^d$ be a continuous injective map. Then for any compact $X \subset \mathfrak{R}^d$,*

$$\text{int } f(X) = f(\text{int } X) \quad \text{and} \quad \partial f(X) = f(\partial X).$$

Proof. First we prove the simple inclusion $\text{int } f(X) \subseteq f(\text{int } X)$. Taking a point $y \in \text{int } f(X)$ there is a neighbourhood of y , $N_y \subset \text{int } f(X)$. Since f is injective $f^{-1}(N_y) \subset X$, and since f is continuous, this set forms an open neighbourhood of $f^{-1}(y)$: it follows that $f^{-1}(y) \in \text{int } X$. [Note that this result holds for any continuous map on any topological space.]

To prove the opposite inclusion we use the Invariance of Domain: if $y = f(x)$ with $x \in \text{int } X$ consider a neighbourhood N_x of X that lies entirely within $\text{int } X$. Brouwer's result now guarantees that $f(N_x)$ is open, and it is clearly contained in $f(X)$. It follows that $y \in \text{int } f(X)$, and so $\text{int } f(X) = f(\text{int } X)$.

The result on boundaries now follows using the identity $\partial Z = Z \setminus (\text{int } Z)$ valid for any closed set Z (essentially the definition of ∂Z), and the fact that for any homeomorphism f and sets $B \subset A$, $f(A \setminus B) = f(A) \setminus f(B)$. \square

2. The Boundaries of Deterministic Attractors

First we consider the case of deterministic attractors.

In order to motivate our more abstract assumptions, consider a dynamical system generated by a system of autonomous ordinary differential equations on \mathfrak{R}^d , $\dot{x} = f(x)$. Assume that given an initial condition $x(0) = x_0$, there is a unique solution $x(t; x_0)$ that is defined on the maximal interval of existence $t \in (T^*(x_0), \infty)$ for some $T^*(x_0) < 0$ (and perhaps $T^*(x_0) = -\infty$). This is the case, for example, if

^aIf $f: \mathfrak{R}^d \rightarrow \mathfrak{R}^d$ is a continuous injective map and U is an open subset of \mathfrak{R}^d , then $f(U)$ is also open, and $f: U \rightarrow f(U)$ is a homeomorphism, see Proposition 7.4 in Dold [8], for example.

f is locally Lipschitz and solutions do not blow up in any finite positive time (see Hartman [10], for example).

For simplicity later we define a solution operator $S(t): \mathbb{R}^d \rightarrow \mathbb{R}^d$ so that

$$S(t)x_0 = x(t; x_0).$$

Note that under our assumptions $S(t)$ is only defined on the whole of \mathbb{R}^d for $t \geq 0$; for $t < 0$ it has domain

$$D(S(t)) = \{x_0 \in \mathbb{R}^d: T^*(x_0) < t\}.$$

With this minor caveat, the following properties of $S(t)$ are clear:

- (i) $S(0) = \text{Id}_{\mathbb{R}^d}$,
- (ii) $S(t)S(s)x_0 = S(t+s)x_0$ for all values of t, s , for which the expressions make sense,^b
- (iii) where $S(t)x$ is defined jointly continuous in t and x , and
- (iv) $S(t)$ is injective for all $t \geq 0$ (and in fact on $D(S(t))$ if $t < 0$).

For each $t \geq 0$ the map $S(t)$ naturally induces a map on subsets X of \mathbb{R}^d via the definition

$$S(t)X = \bigcup_{x \in X} S(t)x.$$

The global attractor \mathcal{A} is the maximal compact invariant set or, equivalently, the smallest invariant set that attracts all bounded sets uniformly:

- (i) \mathcal{A} is compact,
- (ii) \mathcal{A} is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and
- (iii) \mathcal{A} attracts all bounded sets, i.e. for any bounded set X

$$\text{dist}(S(t)X, \mathcal{A}) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

where

$$\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} |x - y|$$

is the Hausdorff semi-distance between X and Y .

For more details see Hale [9], Robinson [11], Stuart & Humphries [14], or Temam [16]. It is an important consequence of Lemma 1.1 and the invariance of \mathcal{A} that the boundary of \mathcal{A} is also invariant under the flow,

$$S(t)\partial\mathcal{A} = \partial\mathcal{A} \quad \text{for all } t \in \mathbb{R} \tag{2.1}$$

(cf. Theorem 1.6 in Bhatia & Szegö [2]).

^bMore formally, and unhelpfully, this is

$$s \in (-T^*(x_0), \infty), \quad t \in (-T^*(x_0) - s, \infty).$$

We note here for later use that *on the attractor* the semigroup $S(t)$ becomes a group. Indeed, given any $x \in \mathcal{A}$ and $t > 0$ we can define $S(-t)x$ as follows: since $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$, there exists $x_{-t} \in \mathcal{A}$ such that $S(t)x_{-t} = x$. Such an x_{-t} is well-defined, since $S(t)$ is injective for $t \geq 0$. Furthermore it is clear that

$$S(s)x_{-t} = x_{-(t-s)}$$

for all $0 < s < t$. We can therefore define $S(-t)x = x_{-t}$. Thus, in fact, the invariance of \mathcal{A} (part (ii) above) holds for all $t \in \mathbb{R}$.

The attractor is usually constructed as the omega limit set of some suitable absorbing^c set D , where

$$\begin{aligned} \Omega_D &= \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)D} \\ &= \{x: \text{there are sequences } t_n \rightarrow \infty \text{ and } d_n \in D \text{ with } S(t_n)d_n \rightarrow x\}. \end{aligned}$$

(We use the slightly non-standard notation for the omega limit set to remain consistent with the stochastic case which is considered later.)

We now show that in the deterministic dynamical systems considered here “ ∂ and Ω commute” for compact sets containing the attractor. Clearly this includes the closure of any open ball $B(0, R)$ large enough to contain \mathcal{A} .

Theorem 2.1. *Suppose that D is a compact subset of \mathbb{R}^d such that $\mathcal{A} = \Omega_D$ and D contains \mathcal{A} . Then $\partial\mathcal{A} = \Omega_{\partial D}$.*

This result (with the assumption that D is “compact” replaced by the assumption that it is “closed and bounded”) would be essentially vacuous in an infinite-dimensional space H , since any compact subset K of H would have empty interior and so be equal to its boundary. Thus in such a situation $\partial\mathcal{A} = \Omega_{\partial D}$ is merely a restatement of $\mathcal{A} = \Omega_D$. However, in a more positive direction note that in many interesting dynamical systems the flow contracts volumes; once again the interior of the attractor is empty and in this case we have $\mathcal{A} = \Omega_{\partial\Omega}$.

Proof. First we show that $\partial\mathcal{A} \subset \Omega_{\partial D}$. Given some $x \in \partial\mathcal{A}$, consider the set $S(n)^{-1}B(x, 1/n)$, where $B(x, \rho)$ the ball of radius ρ about x ,

$$B(x, \rho) = \{y \in \mathbb{R}^d: |x - y| < \rho\}.$$

Since $x \in \partial\mathcal{A}$, which is invariant (see (2.1)), $S(n)^{-1}x = S(-n)x \in \partial\mathcal{A}$. Since $S(n)$ is continuous, $S(n)^{-1}B(x, 1/n)$ is an open set; it is nonempty since it contains $S(-n)x$, and hence it forms a neighbourhood of $S(-n)x$. Since $S(-n)x \in \partial\mathcal{A}$, either (i)

$$S(n)^{-1}B(x, 1/n) \cap [D \setminus \mathcal{A}] = \emptyset$$

(this cannot happen if D contains a neighbourhood of \mathcal{A}) or (ii) there exists a point $b_n \in D \setminus \mathcal{A}$ such that $S(n)b_n \in B(x, 1/n)$; the latter case is illustrated in Fig. 1.

^cA set D is said to be absorbing if for any bounded set X , there exists a time t_X such that $S(t)X \subseteq D$ for all $t \geq t_X$.

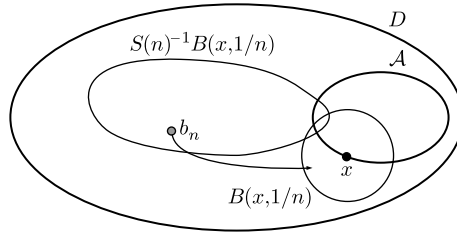


Fig. 1. Given an $x \in \partial A$ we find points $b_n \in D \setminus A$ such that $S(n)b_n \rightarrow x$ by taking preimages of $B(x, 1/n)$ under $S(n)$.

In case (i) it follows that $S(-n)x \in \partial D$, and we set $t_n = n$ and $d_n^* = S(-n)x$.

Case (ii) requires more work. Since $b_n \notin A$ and D is attracted to A , there is a time $\sigma_n \geq 0$ such that $b_n \notin S(t)D$ for all $t \geq \sigma_n$. It follows that during its evolution the set $S(\cdot)D$ must “pass through” b_n , and hence $b_n \in S(s_n)[\partial D]$ for some s_n , see Fig. 2(a). More formally, set

$$s_n = \inf\{\sigma : b_n \notin S(t)D \text{ for all } t \geq \sigma\}$$

which is finite. It follows that there exist sequences $\tilde{s}_j \uparrow s_n$ and $d_j \in D$ such that $b_n = S(\tilde{s}_j)d_j$; since D is compact there is a subsequence (which we relabel) such that $d_j \rightarrow d_n^* \in D$. Since $S(\cdot)$ is continuous, it follows that $b_n = S(s_n)d_n^*$ for some $d_n^* \in D$.

We now show that in fact $d_n^* \in \partial D$. If not, then $b_n = S(s_n)\beta$ for some $\beta \in \text{int } D$. The continuity of $S(t)\beta$ in t implies that there exists a δ_0 with $0 < \delta_0 < |T^*(\beta)|$ such that $S(\delta)\beta \in \text{int } D$ for all $|\delta| < \delta_0$. It follows that for all $|\delta| < \delta_0$

$$b_n = S(s_n)\beta = S(s_n + \delta)[S(-\delta)\beta] \in S(s_n + \delta)\text{int } D,$$

contradicting the definition of s_n . Thus $b_n = S(s_n)d_n^*$ for some $d_n^* \in \partial D$; set $t_n = n + s_n$.

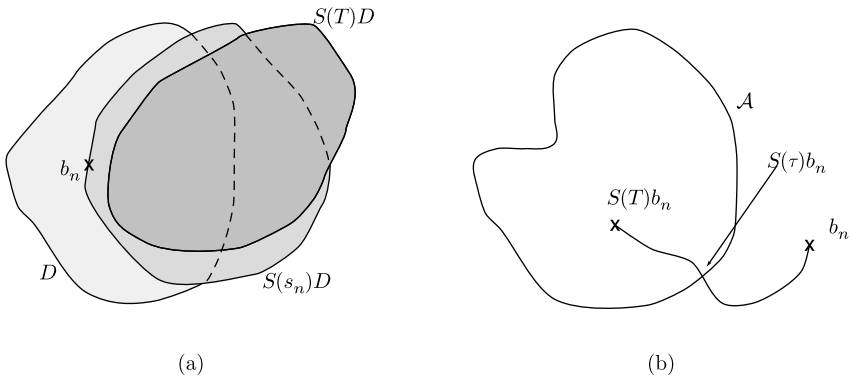


Fig. 2. (a) If $b_n \in D$ and $b_n \notin S(T)D$ for some $T > 0$, then $b_n \in S(s_n)[\partial D]$ with $s_n \in (0, T)$. (b) If $b_n \notin A$ and $S(T)b_n \in \text{int } A$, then $S(\tau)b_n \in \partial A$ for some $0 < \tau < T$.

We now have a sequence t_n such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$, and since either (i) $S(t_n)d_n^* = x$ or (ii) $S(t_n)d_n^* = S(n + s_n)d_n^* = S(n)b_n \in B(x, 1/n)$, it is clear that

$$\lim_{n \rightarrow \infty} S(t_n)d_n^* = x,$$

and it follows that $x \in \Omega_{\partial D}$.

It remains to show the reverse inclusion, $\partial \mathcal{A} \supset \Omega_{\partial D}$. Suppose for a contradiction that $x \in \Omega_{\partial D} \cap \text{int } \mathcal{A}$. Since $x \in \text{int } \mathcal{A}$, we have $B(x, \delta) \subset \text{int } \mathcal{A}$ for some $\delta > 0$. However, since $x \in \Omega_{\partial D}$ there exist sequences $b_n \in \partial D$ and $t_n \rightarrow \infty$ such that $S(t_n)b_n \rightarrow x$. Choosing n large enough that $|S(t_n)b_n - x| < \delta$ implies that $S(t_n)b_n \in \text{int } \mathcal{A}$.

Since $b_n \in \partial D$ then either $b_n \in \partial \mathcal{A}$ or $b_n \notin \mathcal{A}$. In the former case we immediately obtain a contradiction from the fact that $\partial \mathcal{A}$ is invariant; while if $b_n \notin \mathcal{A}$ the trajectory $S(\cdot)b_n$ starts in ∂D but moves continuously into the interior of $\partial \mathcal{A}$. It is clear that if

$$\tau = \sup\{t: S(t)b_n \notin \mathcal{A} \text{ for all } t \leq \tau\}$$

(this is clearly finite and, since \mathcal{A} is compact and $S(t)b_n$ continuous, strictly positive), then $S(\tau)b_n \in \partial \mathcal{A}$, since $\mathbb{R}^d \setminus \partial \mathcal{A}$ is open and $S(t)b_n$ is continuous (see Fig. 2(b)). That subsequently $S(t_n)b_n \in \text{int } \mathcal{A}$ contradicts the invariance of $\partial \mathcal{A}$, and the proof is complete. \square

An interesting corollary of this result concerns “discrete-time Ω -limit sets”, which are useful in the context of numerical calculations. We define

$$\begin{aligned} \Omega_D(\Delta t) := \{x \in X \mid \text{there exist integers } n_j \rightarrow \infty \text{ and } d_j \in D \\ \text{such that } S(n_j \Delta t)d_j \rightarrow x, \text{ as } j \rightarrow \infty\}. \end{aligned} \tag{2.2}$$

Note that $\mathcal{A} = \Omega_D(\Delta t)$, since clearly $\Omega_D(\Delta t) \subseteq \Omega_D$, and, because $D \supseteq \mathcal{A}$ and \mathcal{A} is invariant, $S(n\Delta t)D \supseteq \mathcal{A}$ for all n . A similar result also holds for the boundary of \mathcal{A} :

Proposition 2.1. *Let the assumptions of Theorem 2.1 hold, and in addition assume that D consists of a single connected component. Then*

$$\Omega_{\partial D}(\Delta t) = \partial \mathcal{A} \quad \text{for any } \Delta t > 0.$$

Proof. It is clear that $\Omega_{\partial D}(\Delta t) \subseteq \Omega_{\partial D} = \partial \mathcal{A}$. For the reverse inclusion choose $x \in \partial \mathcal{A} = \Omega_{\partial D}$ and suppose that $x \notin \Omega_{\partial D}(\Delta t)$. Then there exist $\delta > 0$ and $N \in \mathbb{N}$ such that

$$S(n\Delta t)[\partial D] \cap B(x, \delta) = \emptyset \quad \text{for all } n \geq N. \tag{2.3}$$

Since D consists of a single connected component so does $S(n_j \Delta t)$, and so it follows from (2.3) that for any sequence $n_j \rightarrow \infty$, either

$$S(n_j \Delta t)D \subseteq B(x, \delta) \tag{2.4}$$

or

$$S(n_j \Delta t)D \cap B(x, \delta) = \emptyset$$

for all j such that $n_j \geq N$. Since $D \supseteq \mathcal{A}$, invariance of \mathcal{A} implies $S(n_j \Delta t)D \supseteq \partial \mathcal{A}$ for all j and so (2.4) must hold for all j large enough. But then $B(x, \delta) \subseteq \mathcal{A} = \Omega_D$ which contradicts the definition of boundary since by Theorem 2.1, $x \in \Omega_{\partial D} = \partial \mathcal{A}$. Therefore $x \in \Omega_{\partial D}(\Delta t)$. □

3. The Boundaries of Random Attractors

In the random case we do not, in general, have an equality of the form $\Omega_{\partial D} = \partial \mathcal{A}$ (for some “suitable” deterministic D), but only $\partial \mathcal{A} \subseteq \Omega_{\partial D}$. However, for a large class of systems (which we would expect to include all those with sufficiently rich additive noise), we have the stronger and perhaps surprising equality $\Omega_{\partial D} = \mathcal{A}$.

Just as in the previous section, our assumptions are motivated by a particular example, namely the cocycles arising from stochastic ordinary differential equations. However, we once again state our result in a more abstract framework.

Given any deterministic initial condition we assume that solutions exist for all positive times, and for some (possibly small) time interval into the past (which may depend on the particular realisation of the noise). Note that our system is a hybrid of a local random dynamical system (existence only assured on some time interval containing zero) and a one-sided RDS, see Chap. One in Arnold [1] for more details.

A random dynamical system consists of a base flow, “the noise”, and a cocycle dynamical system on the “physical” phase space. More formally, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (Ω, θ) a two-sided, measurable dynamical system that preserves the measure \mathbb{P} ; furthermore we assume that (θ, \mathbb{P}) is ergodic. (Ergodicity of θ , while true in most applications, is not usually part of the formal definition of an RDS.)

We assume that $\phi: \mathbb{R} \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies

- (i) $\phi(0, \omega) = \text{Id}_{\mathbb{R}^d}$,
- (ii) ϕ satisfies the cocycle property,

$$\phi(t, \theta_s \omega) \phi(s, \omega) x_0 = \phi(t + s, \omega) x_0, \tag{3.1}$$

\mathbb{P} -a.s. for all t, s , where the expressions make sense,

- (iii) $\phi(t, \omega)x$ is continuous in both t and x , and
- (iv) ϕ is a $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ -measurable map, where $\mathcal{B}(X)$ is the Borel sigma algebra of X .

From conditions (ii) and (iii), $\phi(t, \omega)$ is a continuous injective map where defined. (Our ϕ is an injective, local left-sided random dynamical system; we will refer to it as an “injective random dynamical system” for short.)

Attractors for random dynamical systems were introduced by Crauel & Flandoli [4] and Schmalfuss [12], with notable developments by Crauel, Desbussche, & Flandoli [3] and Schenk-Hoppé [13]:

Definition 3.1. The random global attractor $\mathcal{A}(\omega)$ for the RDS ϕ is a \mathbb{P} -almost surely random compact set^d that also satisfies

- (i) Invariance: for all $t \geq 0$, $\phi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$ \mathbb{P} -a.s.
- (ii) Attraction: for any bounded deterministic set $B \subset \mathfrak{R}^d$, \mathbb{P} -a.s.

$$\text{dist}(\phi(t, \theta_{-t}\omega)B, \mathcal{A}(\omega)) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \tag{3.2}$$

Crauel [6] showed that the random attractor is \mathbb{P} -a.s. unique and characterised by the above properties. Furthermore it is known, see Crauel & Flandoli [4], that for an RDS with a random compact absorbing^e set, the global attractor is given by

$$\mathcal{A}(\omega) = \overline{\bigcup_{B \subset \mathbb{R}^d} \Omega_B(\omega)}, \tag{3.3}$$

where the union is taken over all bounded subsets B of \mathbb{R}^d , and now the omega-limit set is defined as

$$\begin{aligned} \Omega_B(\omega) &= \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \phi(s, \theta_{-s}\omega)B} \\ &= \{x \in X: \text{there exists } t_n \rightarrow \infty \text{ and } d_n \in B \\ &\quad \text{such that } \phi(t_n, \theta_{-t_n}\omega)d_n \rightarrow x \text{ as } n \rightarrow \infty\}. \end{aligned}$$

Note that it follows from Lemma 1.1 that the boundary of the random attractor is invariant,

$$\phi(t, \omega)\partial\mathcal{A}(\omega) = \partial\mathcal{A}(\theta_t\omega) \quad \text{for all } t \in \mathfrak{R}, \tag{3.4}$$

since $\phi(t, \omega)$ is an injective continuous map for each (t, ω) with $t \geq 0$ (see also Schenk-Hoppé [13]; we note here that [13] also shows that $\partial\mathcal{A}(\omega)$ has many properties in common with $\mathcal{A}(\omega)$ itself: it is a measurable compact random set, and under certain conditions supports its own invariant measures).

The proof that $\partial\mathcal{A} = \Omega_{\partial D}$ in the random case is similar to the proof of Theorem 2.1 but has some subtleties, since now we cannot expect to find a single deterministic set such that $\mathcal{A}(\omega) \subset D$ for all ω . Indeed, the standard proof of the existence of the random attractor constructs it as in (3.3). However, we can use a result due to Crauel [6] to find a single deterministic bounded set whose Ω -limit set gives the attractor (this is the first part of Eq. (3.6) in the statement of the theorem).

^dA map $K: \Omega \rightarrow X$ is called a compact random set if $K(\omega)$ is compact and for all $x \in X$ $\omega \rightarrow \text{dist}(x, K(\omega))$ is \mathbb{P} -almost surely measurable.

^eIn this case the definition involves absorption “in the pullback sense”: $K(\omega)$ is absorbing if for all bounded deterministic sets X there exists a random time $t_X(\omega) \geq 0$ such that

$$\phi(t, \theta_{-t}\omega)X \subset K(\omega) \quad \text{for all } t \geq t_X(\omega).$$

Theorem 3.1. *Suppose that $\mathcal{A}(\omega)$ is the random attractor for an injective RDS with ergodic base flow θ_\cdot , and D is a deterministic set such that*

$$\mathbb{P}(\mathcal{A}(\cdot) \subset D) > 0. \tag{3.5}$$

Then \mathbb{P} -a.s.

$$\mathcal{A}(\omega) = \Omega_D(\omega) \quad \text{and} \quad \partial\mathcal{A}(\omega) \subseteq \Omega_{\partial D}(\omega). \tag{3.6}$$

If we also have

$$\mathbb{P}(\mathcal{A}(\cdot) \cap D = \emptyset) > 0, \tag{3.7}$$

then

$$\Omega_{\partial D}(\omega) = \mathcal{A}(\omega).$$

Proof. Crauel [6] (see also Crauel *et al.* [3]) showed that when θ_\cdot is ergodic the condition

$$\mathbb{P}(\mathcal{A}(\cdot) \subset D) > 0$$

implies that \mathbb{P} -a.s.

$$\mathcal{A}(\omega) = \Omega_D(\omega),$$

which is the first half of (3.6).

In order to prove the boundary result, first choose one of the ω such that $\mathcal{A}(\omega) = \Omega_D(\omega)$. A simple application of the Poincaré recurrence theorem (see e.g. Theorem 1.4 in Walters [17]) to the ergodic dynamical system $(\{\Omega, \mathcal{F}, \mathbb{P}\}, \theta_{-1})$, guarantees that

$$\text{there exist } t_n \rightarrow \infty \quad \text{such that } \mathcal{A}(\theta_{-t_n}\omega) \subset D. \tag{3.8}$$

Indeed, let

$$E = \{\omega \in \Omega: \mathcal{A}(\omega) \subset D\}.$$

Then by assumption $\mathbb{P}(E) > 0$, and hence the iterates of almost all points $\omega \in E$ under θ_{-1} return to E infinitely often (this is the Poincaré recurrence theorem). Now consider the set

$$E_\infty = \{\omega \in \Omega: \text{points on the orbit of } \omega \text{ under } \theta_{-1} \text{ return to } E \text{ infinitely often}\}.$$

Then E_∞ is an invariant set under θ_{-1} ; since θ_{-1} is ergodic it follows that $\mathbb{P}(E_\infty)$ is either zero or one, and since we already know that $\mathbb{P}(E_\infty) \geq \mathbb{P}(E) > 0$, we must have $\mathbb{P}(E_\infty) = 1$, i.e. (3.8) holds \mathbb{P} -a.s. (the sequence t_n may depend on ω , but this is not a problem).

Now choose $x \in \partial\mathcal{A}(\omega)$. Arguing as before, consider $\phi(t_n, \theta_{-t_n})^{-1}x = \phi(-t_n, \omega)x$, which is an element of $\partial\mathcal{A}(\theta_{-t_n}\omega)$ since $\partial\mathcal{A}(\cdot)$ is invariant (3.4). Since $\phi(t_n, \theta_{-t_n})$ is continuous, $\phi(t_n, \theta_{-t_n}\omega)^{-1}B(x, 1/n)$ contains a neighbourhood of $\phi(-t_n, \omega)x$, and hence we can either (i) conclude directly that there exists a point

$d_n \in \partial D$ such that $\phi(\tau_n, \theta_{-\tau_n}\omega)d_n = x$, and set $\tau_n = t_n$; or (ii) that there exists a point $b_n \in D \setminus \mathcal{A}(\theta_{-t_n}\omega)$ such that

$$|\phi(t_n, \theta_{-t_n}\omega)b_n - x| \leq 1/n. \tag{3.9}$$

For case (ii), since $\mathcal{A}(\theta_{-t_n}\omega)$ attracts D in the pullback sense (3.2), there exists some σ_n such that

$$b_n \notin \phi(s, \theta_{-t_n-s}\omega)D$$

for all $s \geq \sigma_n$. Define

$$s_n = \inf\{\sigma: b_n \notin \phi(s, \theta_{-t_n-s}\omega)D \text{ for all } s \geq \sigma\}.$$

Arguing as in the deterministic case, the compactness of D and the continuity of $\phi(s, \theta_{-t_n-s}\omega)$ in s implies that

$$b_n = \phi(s_n, \theta_{-t_n-s_n}\omega)d_n$$

for some $d_n \in D$.

If $d_n \notin \partial D$, but $d_n = \beta \in \text{int } D$, we now use the continuity of $\phi(s, \theta_{-t_n-s}\omega)\beta$ to find a $\delta_0 > 0$ such that for all $|\delta| < \delta_0$ we still have

$$\phi(\delta, \theta_{-t_n-s_n}\omega)\beta \in \text{int } D$$

(and the left-hand side is defined). It follows that

$$\begin{aligned} b_n &= \phi(s_n, \theta_{-t_n-s_n}\omega)\beta \\ &= \phi(s_n + \delta, \theta_{-t_n-(s_n+\delta\omega)})[\phi(-\delta, \theta_{-t_n-s_n}\omega)\beta] \in \phi(s_n + \delta, \theta_{-t_n-(s_n+\delta\omega)})\text{int } D \end{aligned}$$

for all δ with $|\delta| < \delta_0$, contradicting the definition of s_n . Thus

$$b_n = \phi(s_n, \theta_{-t_n-s_n}\omega)d_n$$

with $d_n \in \partial D$.

Now we use the cocycle property (3.1): we have

$$\begin{aligned} \phi(t_n, \theta_{-t_n}\omega)b_n &= \phi(t_n, \theta_{-t_n}\omega)\phi(s_n, \theta_{-s_n-t_n}\omega)d_n \\ &= \phi(t_n + s_n, \theta_{-t_n-s_n}\omega)d_n, \end{aligned}$$

and we set $\tau_n = t_n + s_n$.

We now have $d_n \in \partial D$ and $\tau_n \rightarrow \infty$ such that

$$|\phi(\tau_n, \theta_{-\tau_n}\omega)d_n - x| \leq 1/n,$$

and so $x \in \Omega_{\partial D}(\omega)$ as required.

Under the additional assumption (3.7), we can use the Poincaré recurrence theorem once more to show that for almost every $\omega \in \Omega$ there exists a sequence of times $\tau_n \rightarrow \infty$ such that

$$\mathcal{A}(\theta_{-\tau_n}\omega) \cap D = \emptyset.$$

By taking subsequences if necessary, we can interleave $\{\tau_n\}$ and $\{t_n\}$, so that $\tau_n \leq t_n \leq \tau_{n+1}$. In particular,

$$\mathcal{A}(\theta_{-\tau_{n+1}}\omega) \cap D = \emptyset \quad \text{but} \quad \mathcal{A}(\theta_{-t_n}\omega) \subseteq D.$$

Now fix ω and take any $x \in \mathcal{A}(\omega)$. For each n , let $x_n \in \mathcal{A}(\theta_{-\tau_{n+1}})$ be the point such that

$$\phi(\tau_{n+1}, \theta_{-\tau_{n+1}}\omega)x_n = x$$

(such a point exists due to the invariance of $\mathcal{A}(\cdot)$). Now, since $x_n \notin D$ and $\phi(\tau_{n+1} - t_n, \theta_{-\tau_{n+1}}\omega)x_n \in D$, there exists an s_n with $0 < s_n < \tau_{n+1} - t_n$ such that

$$d_n := \phi(s_n, \theta_{-\tau_{n+1}}\omega)x_n \in \partial D.$$

Thus

$$x = \phi(\tau_{n+1} - s_n, \theta_{-\tau_{n+1}+s_n}\omega) d_n.$$

Since $t_n < \tau_{n+1} - s_n < \tau_{n+1}$, it follows that $x \in \Omega_{\partial D}(\omega)$ as claimed. □

We note that condition (3.7) fails when

$$\mathbb{P}(\mathcal{A} \cap D) = 1.$$

For equations with multiplicative noise, this is not unusual. Indeed, much of the theory of bifurcations for stochastic equations (see Chap. 8 of Arnold [1], or Crauel, Imkeller, & Steinkamp [5], for example) has been developed for models

$$dx = f(x)dt + \sigma(x) \circ dW_t,$$

where $f(x_0) = \sigma(x_0) = 0$ for some $x_0 \in \mathbb{R}^d$, and in this case $x_0 \in \mathcal{A}(\omega)$ for all ω .

However, for equations with an additive noise we would expect that

$$\mathbb{P}(\mathcal{A} \cap D) < 1 \tag{3.10}$$

for any bounded set D . For the particular example equation

$$dx = f(x) + \sigma dW_t,$$

with $f: \mathbb{R}^d \rightarrow \mathbb{R}^d$ bounded, the argument in the proof of Theorem 4.6 in Crauel [7] can easily be adapted to show that (3.10) holds.

4. Conclusions

We have shown that the boundaries of globally attracting sets are given as the omega-limit sets of the boundaries of suitably large sets. In particular this enables us to consider the evolution of $(d - 1)$ -dimensional sets in order to pinpoint the attractor for a system of differential equations in \mathbb{R}^d .

This observation can be used as the basis for an efficient numerical scheme to compute attractors in planar (and perhaps three-dimensional) systems. Such an algorithm is described and implemented by one of the authors in a forthcoming paper [15].

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